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The stability of a meniscus joining a vertical rod to a bath of liquid

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A circular cylinder is held vertically and withdrawn from a large bath of liquid. When the horizontal end face of the cylinder rises above the mean undisturbed liquid level, an axially symmetrical meniscus is formed, which joins the cylinder at the circumference of the end. If the cylinder is further withdrawn, at a certain height the meniscus breaks. By means of the calculus of variations the condition for stability is derived in terms of the increase in height and the increase in the angle between the tangent to the meniscus and the face of the cylinder at the point where the meniscus joins it. The meniscus is unstable as long as these changes are of the same sign.

1. Introduction

The dominant influence of surface-tension forces is apparent in many phenomena and often they have important technological effects. This has stimulated a growth of interest in the shapes of liquid surfaces, particularly since the derivation of values of surface tension from experiment usually requires knowledge of the solutions of the fundamental equation of equilibrium. This equation has therefore often been the subject of numerical studies (see for instance Princen 1969). Insight into another aspect of these solutions has recently been obtained from the examination of stability criteria by Padday & Pitt (1973) using numerical methods and by Pitts (1973, 1974) using the variational calculus. From this work has come the suggestion of new methods which offer advantages in the measurement of surface tension (Padday & Pitt 1975; Pitts 1975).

Padday & Pitt proposed that surface tension could be accurately determined in the following way. If a circular cylinder is held vertically and withdrawn from a large bath of liquid, then as the horizontal end of the cylinder rises above the mean horizontal free liquid surface, a meniscus is formed whose section is similar to that in figure 1. The meniscus is rotationally symmetrical about the axis OA of the cylinder; the narrowing of the meniscus shown in the figure is of course not always present. Padday & Pitt made the important remark that, since the upward force needed to lift the cylinder passes through a maximum before the meniscus breaks, the measurement of this maximum force could be performed with great accuracy, since it would be possible to make observations while passing through the maximum. They calculated the maximum force in terms of the dimensions of the cylinder, the surface tension and the density of the liquid,

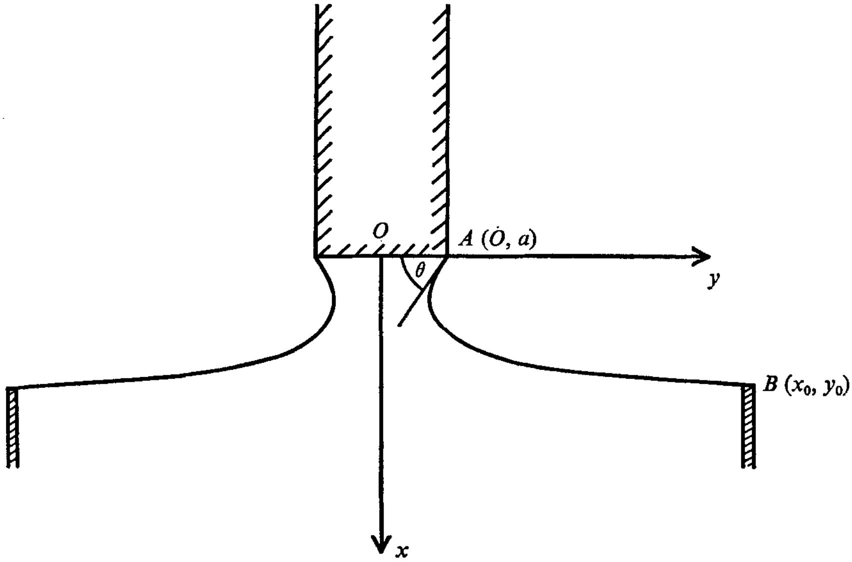

FIGURE 1. The axially symmetrical meniscus and the co-ordinate system. The tangent to the meniscus at A makes an angle θ with the horizontal face of the withdrawn cylinder. B represents the top of the wall of the dish of liquid.

and hence were able to derive values of surface tension from their experiments.

Kovitz (1975) has independently proposed a method closely related to this. He suggested that the surface tension could be derived from the observation of the maximum height to which the cylinder could be withdrawn before the meniscus breaks. The writer's experiments (Pitts 1975), in which the maximum height of a drop hanging from a tube is used to find the surface tension, shows that such a method based on the determination of the point of transition from stable to unstable equilibria can indeed be successfully used.

The arrangement shown in figure 1 is also encountered in the growth of crystals from a melt. It is often highly desirable that the solidified material, which forms the withdrawing cylinder, should have a uniform cross-section. It is also obviously necessary to avoid breaking the meniscus, and the shape of the liquid surface is therefore of great importance.

With these examples in mind it is of interest to examine by means of the calculus of variations the stability of menisci like those in figure 1. There does not appear to be any published treatment of this problem apart from the suggestive numerical study by Padday & Pitt (1973).

A good model of the actual configuration would appear to be one in which the liquid surface extends to infinity, so that the problem is physically determined when the height of the withdrawing cylinder above the mean liquid surface is given. However, there are difficulties in applying many of the standard results in the calculus of variations to an infinite surface, if rigour is demanded.

To avoid a discussion of purely mathematical aspects of the calculus of variations, we can instead return to the discussion of the actual system, in which of course the surface is finite. We now encounter a different complication, which arises from the physics of the problem. Suppose we imagine that the liquid surface extends to the wall of a circular dish whose centre lies on the extended axis of the vertical cylinder. We suppose that B in figure 1 represents the top of the wall of the dish. It will be convenient to refer to the co-ordinate axes shown in figure 1; the abscissa is downward along the axis of the cylinder, whose radius is a, and the ordinate is in the horizontal face of the cylinder, from the perimeter of which the meniscus hangs. Then knowledge of the height x_0 from the base of the cylinder to the horizontal level B of the top of the dish and the value of the radius y_0 of the dish (together with the values of surface tension, gravity and density) no longer defines a unique meniscus. We must know in addition either the total volume of liquid present or the value of the pressure in the liquid at some point.

That this is so can be seen by considering the situation when the dish is not much wider than the cylinder. We could imagine that the liquid in the dish initially has a surface which is lowest in the centre and which rises to the top of the wall B as the side of the dish is approached. If the cylinder is brought vertically down to touch the liquid, then raised again to a certain height above the level B, a particular meniscus will be formed. However, the liquid initially in the dish could have been of sufficient volume that its height on the axis of symmetry exceeded that of the level B, surface tension preventing its overflowing. Then if the vertical cylinder were brought into contact and withdrawn to the same height as before, an entirely different meniscus would result. Furthermore, as the size of the dish becomes very large, these two configurations remain different near the walls of the dish.

This example, in which the volume of liquid is prescribed, would therefore lead to a variational problem with a subsidiary constraint. Again, if rigour is demanded, substantial difficulties arise, not least in considering the limiting case of very large surfaces, which is the one of most practical importance.

A conditional variational problem can be avoided, at the cost of some artificiality, if we suppose that at a particular place, which can conveniently be chosen as B in figure 1, the tangent to the liquid surface is horizontal and the pressure in the liquid has a given value. This is equivalent to specifying the radius of curvature (in the vertical plane) of the profile of the surface at B. We then consider what happens when B is removed to large distances.

In our discussion of the variational problem only axially symmetrical perturbations will be considered. The results obtained provide a theoretical basis for the calculations of Padday & Pitt and clarify some of the observations made by Kovitz.

2. Equilibrium

As previously explained, we suppose that the liquid surface is horizontal at $B(x_0, y_0)$ and that the pressure in the liquid there is p_0. We require an expression for the energy of the liquid as the basis for our variational treatment. We take the level B as the reference for potential energy.

If γ is the surface tension and ds is an element of length along the profile, then the surface energy required in the formation of an elementary slice of liquid of thickness dx is $2\pi\gamma y\, ds$. The pressure in the liquid at the slice is $p_0 + \rho g(x - x_0)$, where ρ is the density of the liquid and g is the acceleration due to gravity. The slice of liquid therefore requires the expenditure of the amount of work

$$-\pi y^2[p_0+\rho g(x-x_0)]\,dx+2\pi\gamma y\,ds$$

in its formation. Hence the total energy of the liquid above the level B is

$$E_0=\int_0^{x_0}[2\pi\gamma y(1+y_x^2)^{\frac{1}{2}}-\pi y^2(p_0-\rho g x_0+\rho g x)]\,dx,$$

where $y_x \equiv dy/dx$. We introduce dimensionless units by writing

$$l=(\gamma/\rho g)^{\frac{1}{2}},\quad E_0=\pi\gamma l^2 E,\quad \rho g l\mu=\rho g x_0-p_0,$$
$$l\xi=x,\quad l\eta=y,\quad l\kappa=x_0,\quad l\eta_0=y_0,\quad l\lambda=a.$$

Then
$$E=\int_0^{\kappa}F(\xi,\eta,\eta_\xi)\,d\xi, \tag{1a}$$
where
$$F=2\eta(1+\eta_\xi^2)^{\frac{1}{2}}+\eta^2(\mu-\xi). \tag{1b}$$

We have now to consider the variation of E. Our choice of boundary conditions ensures that standard results in the calculus of variations can be used, since we have a finite surface whose end points A and B are fixed. The Euler–Lagrange equation for the extremals is the well-known expression,

$$\mu-\xi=-\eta^{-1}(1+\eta_\xi^2)^{-\frac{1}{2}}+\eta_{\xi\xi}(1+\eta_\xi^2)^{-\frac{3}{2}}, \tag{2a}$$

which may also be written as

$$\mu-\xi=-\eta\eta_\xi\,d[\eta(1+\eta_\xi^2)^{-\frac{1}{2}}]/d\xi. \tag{2b}$$

When $\xi=\kappa$ in (2a) the quantity $\kappa-\mu$, which corresponds to the (dimensionless) pressure in the liquid at B, is equal to the sum of the curvatures with their appropriate sign, in accordance with elementary physical considerations.

Furthermore, for surfaces like that in figure 1 for which η_ξ increases monotonically from A to B, it follows that $\eta_{\xi\xi}$ is positive and hence $\mu \geqslant \xi$ throughout the curve AB.

If we multiply both sides of (2b) by $\eta\eta_\xi$ and integrate, we readily obtain an expression for the volume v of liquid in the meniscus above the level B, viz.

$$v=\int_0^{\kappa}\eta^2 d\xi=\lambda^2\mu+2\lambda\sin\theta+\eta_0^2(\kappa-\mu), \tag{3}$$

where θ is the angle between the tangent to the meniscus at A and the horizontal face of the cylinder (see figure 1).

The profile cannot possess a corner, i.e. a place at which the tangent experiences a discontinuity. This physically obvious feature is formally proved by the Weierstress–Erdmann conditions (see Bolza 1961, p. 38). If m_1 and m_2 are the gradients adjacent to a corner, we must have

$$F_1(\xi,\eta,m_1)=F_1(\xi,\eta,m_2),\quad (F-\eta_\xi F_1)_{\xi,\eta,m_1}=(F-\eta_\xi F_1)_{\xi,\eta,m_2}, \tag{4),(5}$$

where $F_1\equiv \partial F/\partial\eta_\xi$. Condition (4) gives

$$m_1(1+m_1^2)^{-\frac{1}{2}}=m_2(1+m_2^2)^{-\frac{1}{2}},$$

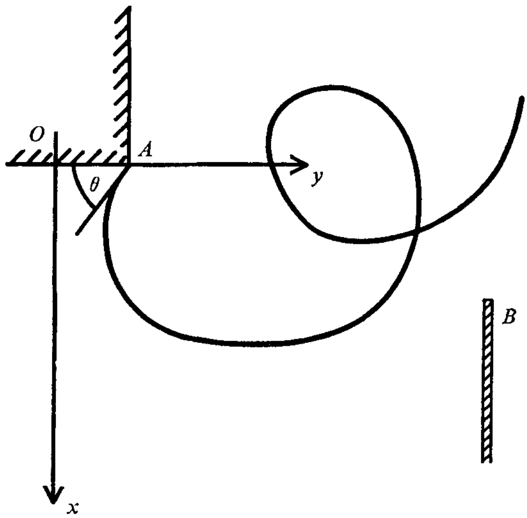

FIGURE 2. Schematic representation of a typical solution of the equation of equilibrium for arbitrary values of the parameters μ and θ.

and after simplification (5) gives

$$(1+m_1^2)^{-\frac{1}{2}} = (1+m_2^2)^{-\frac{1}{2}}.$$

Thus m_1 is equal to m_2 and the gradient is continuous, so that no corner can exist.

Before discussing the stability of the meniscus, it will be helpful to draw attention to some of its qualitative features. These can be seen by examining numerically computed solutions of the equilibrium equation ($2a$). If μ is given together with θ, the solution can be calculated step by step starting from A in figure 1. A typical example is shown diagrammatically in figure 2. In general, the profile so calculated will not pass through B. In order that it may do so for the given θ, a particular value must be given to μ, but now the tangent to the curve at B will not usually be horizontal. This condition can only be met by giving certain values to θ. The calculation of equilibrium profiles is thus a complicated numerical problem.

This behaviour is illustrated by the approximate treatment (given in the appendix) of a meniscus for which the value of θ approaches π. Then the gradient of the surface is everywhere large compared with unity, and explicit expressions can be found which can also be used to demonstrate some of the features of the stability criterion.

3. Stability

We have now to decide whether the curves determined by (2) and the boundary conditions minimize E. Those for which E is not a minimum are unstable, and will not occur in practice.

If the equilibrium curve minimizes E, a number of conditions have to be satisfied (see for instance Bolza 1961, p. 101). We shall consider the requirements when strong variations are allowed. A necessary condition is Legendre's, namely

$$(\partial^2 F/\partial \eta_\xi^2)_{\eta,\eta_\xi} \geqslant 0$$

throughout the curve; i.e.

$$2\eta(1+\eta_\xi^2)^{-\frac{3}{2}} \geqslant 0. \tag{6}$$

This is obviously true as long as η is never negative, which would represent an impossible configuration of the axisymmetrical meniscus. The corresponding sufficient condition for a strong minimum is

$$2\eta(1+p^2)^{-\frac{3}{2}} > 0, \tag{7}$$

which implies $\eta > 0$ throughout the meniscus.

The Weierstrass condition which is necessary for a strong minimum is that the excess function must not be negative, i.e.

$$F(\xi,\eta,p) - F(\xi,\eta,\eta_\xi) - (p-\eta_\xi)\, F_1(\xi,\eta,\eta_\xi) \geqslant 0.$$

This may be written as

$$(1+p^2)^{\frac{1}{2}}(1+\eta_\xi^2)^{\frac{1}{2}} - (1+p\eta_\xi) \geqslant 0,$$

that is

$$(p-\eta_\xi)^2 \geqslant 0, \tag{8}$$

which is always satisfied.

The remaining conditions relate to the zeros of the solutions $u(\xi)$ of the Jacobi accessory equation corresponding to F. After simplification by using ($2a$) this is found to be

$$d(fu_\xi)/d\xi + qu = 0, \tag{9}$$

where

$$f = \eta(1+\eta_\xi^2)^{-\frac{3}{2}}, \quad q = \eta^{-1}(1+\eta_\xi^2)^{-\frac{1}{2}}.$$

The solutions of (9) are given by the partial derivatives of the solutions $\eta(\xi)$ of the Euler equation ($2a$) with respect to their parameters. By differentiation of ($2a$) with respect to θ and some manipulation, a solution of (9) is found to be

$$u_1 = \partial\eta/\partial\theta. \tag{10}$$

Similarly it can be shown that

$$u_2 = \eta_\xi + \partial\eta/\partial\mu \tag{11}$$

is the other solution. The general solution of (9) which vanishes at $A(0,\,\lambda)$ is therefore

$$u = u_1(\xi)\,u_2(0) - u_2(\xi)\,u_1(0).$$

Since near A we have

$$\eta = \lambda - \xi\cot\theta + O(\xi^2), \tag{12}$$

it follows that $u_1(0)$ is zero, and so apart from an arbitrary multiplying factor the solution of the Jacobi equation which vanishes at A is given by (10).

The remaining necessary condition for a strong minimum is that ξ_1, the next zero of $u_1(\xi)$ after A, is greater than or equal to κ; the corresponding sufficient condition is $\xi_1 > \kappa$. We therefore require information about the zeros of u_1, which must be deduced from the differential equations (2) and (9) since explicit solutions cannot be found.

It appears difficult to derive any relevant general results from these equations, but fortunately one important result can be obtained from very simple geo-

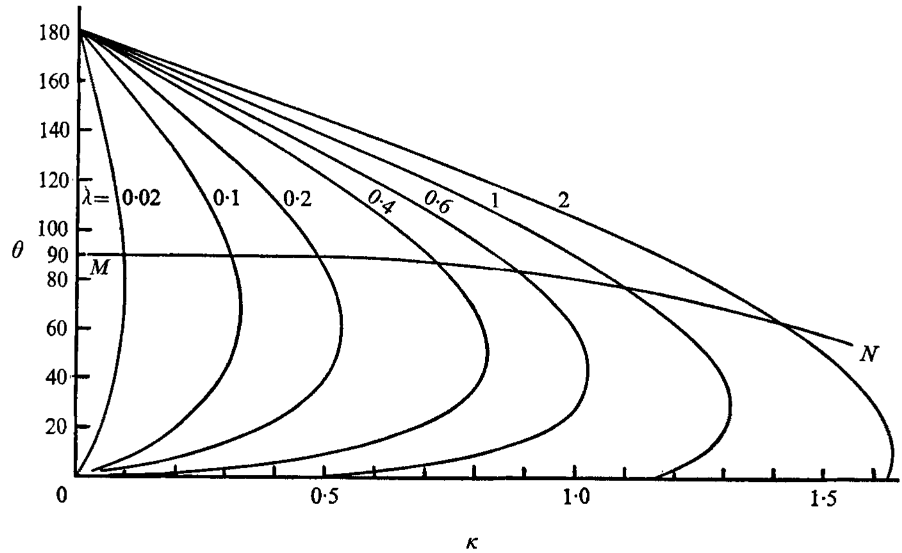

FIGURE 3. The angle θ as a function of meniscus height κ for given values of the cylinder radius λ when the meniscus surface extends to infinity. MN is the locus of the configurations for which the volume between the horizontal through the liquid at infinity and the liquid surface is a maximum.

metrical considerations. From (12) it follows that for small ξ

$$u_1 = \xi \operatorname{cosec}^2 \theta + O(\xi^2), \tag{13}$$

and so u_1 is positive for small positive ξ. We also have the relation

$$(\partial\eta/\partial\theta)_\xi = -\eta_\xi(\partial\xi/\partial\theta)_\eta. \tag{14}$$

Since η_ξ is positive near $B(\kappa, \eta_0)$, then if $(\partial\xi/\partial\theta)_\eta$ is positive here also, u_1 must be negative. Hence for at least one value of ξ between 0 and κ, u_1 has passed through zero. In these circumstances the condition for a minimum is not satisfied, and indeed the second variation of E can be made negative and such a meniscus is unstable.

We can therefore say that, if we have values of μ and θ such that the meniscus approaches the horizontal at $B(\kappa, \eta_0)$ as shown in figure 1, there will be instability if (for the given μ) any small increase in θ would define a new meniscus for which the corresponding value of ξ exceeds κ when $\eta = \eta_0$.

In practice we are interested in the case when B is far removed from the axis of symmetry, and the pressure in the liquid there is very nearly equal to atmospheric pressure. We require information about the stability of these surfaces, and we have at our disposal the criterion just derived in terms of $\partial\eta/\partial\theta$. Now in the limit as B goes to infinity, μ must equal κ and $(2a)$ has a singularity. The angle θ is then the sole parameter of the solutions, and we cannot immediately derive $u_1(\xi)$ for such solutions, because it is the value of $\partial\eta/\partial\theta$ when μ is kept constant. We need an expression for the stability criterion which is related to the solutions of $(2a)$ when μ is equal to κ, which is a function of θ.

The nature of the relationship between θ and κ has been explored by iterated numerical integration of (2a), which allows those values of θ to be found which for given κ yield a solution in which η_0 tends to infinity. The graph in figure 3 shows θ as a function of κ for given values of the radius λ. It will be seen that under certain conditions two values of θ are possible; for example if $\lambda = 0{\cdot}6$ and $\kappa = 0{\cdot}08$, the two values of θ are approximately 9° and 96°.

If μ is regarded as a function of θ, in (2a), we may in principle select the solutions which satisfy the boundary conditions at A and B, and inquire how they change when θ is altered. Regarding μ and θ as distinguishable parameters, the change would be written as

$$\frac{d\eta}{d\theta} = \left(\frac{\partial\eta}{\partial\mu}\right)_\theta \frac{d\mu}{d\theta} + \left(\frac{\partial\eta}{\partial\theta}\right)_\mu,$$

that is, from the definition (10)

$$u_1 = \frac{d\eta}{d\theta} - \left(\frac{\partial\eta}{\partial\mu}\right)_\theta \frac{d\mu}{d\theta}. \tag{15}$$

If for the solution of (2a) when $\mu = \kappa$ we write $\bar{\eta}$, from (15) we expect that the corresponding value of $u_1(\xi)$ will tend to

$$u_1 = d\bar{\eta}/d\theta - \alpha\, d\kappa/d\theta, \tag{16}$$

where

$$\alpha = \lim_{\eta\to\bar{\eta}} (\partial\eta/\partial\mu)_\theta.$$

Now if $d\kappa/d\theta$ is positive it is clear [from the argument using (14)] that, somewhere between A and B, $d\bar{\eta}/d\theta$ must vanish, since from (16) it is readily shown to be positive sufficiently near A and eventually at B is negative. From (2a) we can obtain an approximate expression for the solution η in the vicinity of B, namely

$$\eta = \eta_0 - 2^{\frac{1}{2}} \left(\frac{\kappa-\xi}{\mu-\kappa}\right)^{\frac{1}{2}} + O\left(\frac{\kappa-\xi}{\mu-\kappa}\right), \tag{17}$$

where $\mu > \kappa > \xi$ as already pointed out. From this result it immediately follows that $\partial\eta/\partial\mu$ is positive near B. This result holds however large η_0 may be at B. Thus we may expect that, as η approaches $\bar{\eta}$ ever more closely, both terms in (16) will be negative near B if $d\kappa/d\theta$ is positive. Since u_1 is positive near A, it follows that under these conditions $u_1(\xi)$ will certainly have passed through zero as ξ varies between zero and κ, and so the meniscus will be unstable.

These arguments show that the singular solutions of (2a) which correspond to the infinite meniscus are unstable whenever the values of κ and the corresponding eigenvalue θ increase together. Unfortunately it has not been found possible to prove that the meniscus is stable when $d\kappa/d\theta$ is negative, although experiments by Kovitz and Padday & Pitt show that this is so.

The difficulty in proving stability arises from the absence of explicit expressions for $u_1(\xi)$ and the consequent difficulty in discussing the position of zeros of this function and their dependence on μ and θ. Only the basic Jacobi equation (9) is available. However, in the appendix a short account is given for the case when θ departs by a small amount from its greatest value π. It is shown that

$d\kappa/d\theta$ is negative, and all such solutions are stable, as would be expected from intuitive physical considerations.

4. Discussion

The limitations in the foregoing analysis should be recalled. First of all, axially symmetrical perturbations are the only ones considered. Second, the simple variational approach which has been adopted presupposes that the profile $\eta(\xi)$ is a single-valued function of ξ between the limits A and B, and remains so under variation. A complete discussion of the problem would require the removal of both of these restrictions. Finally, it has only been found possible to prove a criterion for instability, namely that κ and θ increase together. It appears from experiment that the meniscus is stable otherwise. In figure 3, MN shows the locus of the configurations for which the volume of the meniscus is a maximum; it will be seen that they lie outside the region for which $d\kappa/d\theta$ is positive, where the surfaces are unstable. This again is in agreement with experiment.

The stimulus for this investigation is largely due to the work of Dr J. F. Padday and his collaborators in this laboratory, together with interest and helpful comments from others working on related problems, amongst whom I am most grateful to Dr D. H. Michael for his encouragement.

Appendix

If in ($2a$) we suppose that $\eta_\xi \gg 1$, we may take η as the independent variable and, having regard to the sign of the derivative, we obtain

$$\xi - \mu = \eta^{-1} d(\eta d\xi/d\eta)/d\eta, \tag{A 1}$$

which has the solution

$$\xi - \mu = SI_0(\eta) + TK_0(\eta), \tag{A 2}$$

where I_0 and K_0 are modified Bessel functions of order zero. In determining S and T from the boundary conditions we make use of the following well-known properties of the Bessel functions:

$$I_0'(\eta) = I_1(\eta), \quad K_0'(\eta) = -K_1(\eta),$$

$$\eta[I_\nu(\eta) K_\nu'(\eta) - I_\nu'(\eta) K_\nu(\eta)] = -1, \quad \eta[I_\nu(\eta) K_{\nu+1}(\eta) + I_{\nu+1}(\eta) K_\nu(\eta)] = 1,$$

where a prime denotes differentiation with respect to the argument. If we write

$$c = -\cot\theta, \tag{A 3}$$

which is positive when $\frac{1}{2}\pi < \theta < \pi$, then the boundary conditions at A in figure 1 require

$$-\mu = SI_0(\lambda) + TK_0(\lambda), \tag{A 4}$$

$$c = SI_1(\lambda) - TK_1(\lambda), \tag{A 5}$$

from which

$$S = \lambda[cK_0(\lambda) - \mu K_1(\lambda)], \tag{A 6}$$

$$T = -\lambda[cI_0(\lambda) + \mu I_1(\lambda)]. \tag{A 7}$$

		0	0·2	0·4	0·6
$\lambda = 2$	θ (from (A 12))	180	166	154	144
	θ (numerical)	180	166	152	137
$\lambda = 1$	θ (from (A 12))	180	164	150	139
	θ (numerical)	180	164	147	129
$\lambda = 0{\cdot}4$	θ (from (A 12))	180	159	142	130
	θ (numerical)	180	157	134	107

TABLE 1

From these expressions the changes in the solutions which arise when θ (that is, when c) and μ are independently varied can be found.

Since when the surface passes through B the tangent is horizontal, we must have

$$(\partial\xi/\partial\eta)_B = SI_1(\eta_0) - TK_1(\eta_0) = 0. \tag{A 8}$$

Hence from (A 6) and (A 7) we obtain

$$\mu = c[I_1(\eta_0)K_0(\lambda) + I_0(\lambda)K_1(\eta_0)][I_1(\eta_0)K_1(\lambda) - I_1(\lambda)K_1(\eta_0)]^{-1}, \tag{A 9}$$

which gives the relation between c and μ. Finally, the surface must pass through B, and we obtain

$$\xi_0 = c\{I_1(\eta_0)[K_0(\lambda) - K_0(\eta_0)] + K_1(\eta_0)[I_0(\lambda) - I_0(\eta_0)]\}[I_1(\eta_0)K_1(\lambda) - I_1(\lambda)K_1(\eta_0)]^{-1}, \tag{A 10}$$

which determines c.

In the limit when η_0 tends to infinity, we find that

$$\xi - \mu = -cK_0(\eta)K_1^{-1}(\lambda), \tag{A 11}$$

$$\xi_0 = \mu = cK_0(\lambda)K_1^{-1}(\lambda). \tag{A 12}$$

It is of interest to compare the values of θ derived from these expressions for several values of κ ($= \xi_0$) and λ with those calculated numerically. This comparison is given in table 1, where angles have been rounded to the nearest degree.

Finally, we can derive the zero of $u_1(\xi)$ from (14) by considering the zero of $(\partial\xi/\partial c)_\mu$. From (A 2) this will be zero when

$$0 = \frac{\partial S}{\partial c}I_0(\eta) + \frac{\partial T}{\partial c}K_0(\eta),$$

which from (A 6) and (A 7) gives

$$K_0(\lambda)I_0(\eta) = I_0(\lambda)K_0(\eta),$$

that is

$$M(\lambda) = M(\eta), \tag{A 13}$$

where $M(\eta) = I_0(\eta)K_0^{-1}(\eta)$. Since

$$dM(\eta)/d\eta = [\eta K_0^2(\eta)]^{-1} > 0 \tag{A 14}$$

it is obvious that, apart from $\eta = \lambda$, there are no solutions of (A 13). Hence $u_1(\xi)$ has no zero other than at A in figure 1, and the meniscus is stable.

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